変分法

by *T.Koyama*

１．１次元で、変数に $c, x, c' = \partial c / \partial x$ を持つ場合

まず汎関数を以下のように設定する。

$$J = \int \left\{ F\left(c(x), x, \frac{\partial c}{\partial x} \right) \right\} dx \tag{1}$$

第一変分は、

$$\delta J = \left(\lim_{\varepsilon \to 0} \frac{J(c + \varepsilon v) - J(c)}{\varepsilon} \right) \varepsilon \tag{2}$$

さらに、微分が定義できる場合には、

$$\delta J = \left[\frac{\partial J(c + \varepsilon v)}{\partial \varepsilon} \right]_{\varepsilon = 0} \varepsilon \tag{3}$$

にて計算されるので、

$$\begin{aligned}
\delta J &= \left[\frac{\partial}{\partial \varepsilon} \int F\left(c + \varepsilon v, x, \left[\frac{\partial c}{\partial x} \right] + \varepsilon \left[\frac{\partial v}{\partial x} \right] \right) dx \right]_{\varepsilon = 0} \varepsilon = \left[\frac{\partial}{\partial \varepsilon} \int F\left(c + \varepsilon v, x, c' + \varepsilon v' \right) dx \right]_{\varepsilon = 0} \varepsilon \\
&= \left[\int \left[\frac{\partial F}{\partial c} \right] v dx + \int \left[\frac{\partial F}{\partial c'} \right] v' dx \right] \varepsilon \\
&= \int \left[\frac{\partial F}{\partial c} \right] \delta c dx + \int \left[\frac{\partial F}{\partial c'} \right] \delta c' dx
\end{aligned} \tag{4}$$

この第２項を部分積分する。

$$\int \left[\frac{\partial F}{\partial c'} \right] \delta c' dx = \left\{ \left[\frac{\partial F}{\partial c'} \right] \delta c \right\} - \int \frac{d}{dx} \left[\frac{\partial F}{\partial c'} \right] \delta c dx \tag{5}$$

したがって、

$$\begin{aligned}
\delta J &= \int \left[\frac{\partial F}{\partial c} \right] \delta c dx + \int \left[\frac{\partial F}{\partial c'} \right] \delta c' dx \\
&= \int \left[\frac{\partial F}{\partial c} \right] \delta c dx + \left\{ \left[\frac{\partial F}{\partial c'} \right] \delta c \right\} - \int \frac{d}{dx} \left[\frac{\partial F}{\partial c'} \right] \delta c dx \\
&= \left\{ \left[\frac{\partial F}{\partial c'} \right] \delta c \right\} + \int \left(\left[\frac{\partial F}{\partial c} \right] - \frac{d}{dx} \left[\frac{\partial F}{\partial c'} \right] \right) \delta c dx
\end{aligned} \tag{6}$$

これより、オイラー方程式は、

$$\left(\frac{\partial F}{\partial c}\right)-\frac{d}{dx}\left(\frac{\partial F}{\partial c'}\right)=0 \tag{7}$$

なお、この第２項内の$\left(\frac{\partial F}{\partial c'}\right)$は$c, x, \frac{\partial c}{\partial x}$の関数であるので、$x$ で微分する場合には合成関数として扱わなくてはならない。たとえば、

$$\begin{aligned} dF &= \left(\frac{\partial F}{\partial x}\right)(dx)+\left(\frac{\partial F}{\partial c}\right)(dc)+\left(\frac{\partial F}{\partial c'}\right)(dc')=\left(\frac{\partial F}{\partial x}\right)(dx)+\left(\frac{\partial F}{\partial c}\right)(dc)+\left(\frac{\partial F}{\partial c'}\right)\{d(c')\} \\ \frac{dF}{dx} &= \left(\frac{\partial F}{\partial x}\right)+\left(\frac{\partial F}{\partial c}\right)\left(\frac{dc}{dx}\right)+\left(\frac{\partial F}{\partial c'}\right)\left\{\frac{d}{dx}\left(\frac{dc}{dx}\right)\right\}=\left(\frac{\partial F}{\partial x}\right)+\left(\frac{\partial F}{\partial c}\right)c'+\left(\frac{\partial F}{\partial c'}\right)c'' \end{aligned} \tag{8}$$

であるから、同様にして、

$$\frac{d}{dx}\left(\frac{\partial F}{\partial c'}\right)=\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial F}{\partial c'}\right)+\left(\frac{\partial}{\partial c}\right)\left(\frac{\partial F}{\partial c'}\right)c'+\left(\frac{\partial}{\partial c'}\right)\left(\frac{\partial F}{\partial c'}\right)c'' \tag{9}$$

が得られる。

２．２次元で、変数に$c(x, y), x, y, c_x=\partial c/\partial x, c_y=\partial c/\partial y$を持つ場合

まず汎関数を以下のように設定する。

$$J=\iint\left[F\left\{c(x, y), x, y, \frac{\partial c}{\partial x}, \frac{\partial c}{\partial y}\right\}\right]dxdy \tag{10}$$

第一変分は、

$$\delta J=\left\{\lim_{\varepsilon\to 0}\frac{J\{c(x, y)+\varepsilon v(x, y)\}-J\{c(x, y)\}}{\varepsilon}\right\}\varepsilon \tag{11}$$

にて計算されるので、

$$\begin{aligned} \delta J &= \lim_{\varepsilon\to 0}\left[\frac{1}{\varepsilon}\left\{\iint F(c+\varepsilon v, x, y, c_x+\varepsilon v_x, c_y+\varepsilon v_y)dxdy-\iint F(c, x, y, c_x, c_y)dxdy\right\}\right]\varepsilon \\ &= \left(\iint\left(\frac{\partial F}{\partial c}\right)vdxdy+\iint\left(\frac{\partial F}{\partial c_x}\right)v_x dxdy+\iint\left(\frac{\partial F}{\partial c_y}\right)v_y dxdy\right)\varepsilon \\ &= \iint\left(\frac{\partial F}{\partial c}\right)\delta c dxdy+\iint\left(\frac{\partial F}{\partial c_x}\right)\delta c_x dxdy+\iint\left(\frac{\partial F}{\partial c_y}\right)\delta c_y dxdy \end{aligned} \tag{12}$$

この第２および３項を部分積分する。

$$\iint \left(\frac{\partial F}{\partial c_x} \right) \delta c_x dxdy = \int \left\{ \int \left(\frac{\partial F}{\partial c_x} \right) \delta c_x dx \right\} dy$$

$$= \int \left\{ \left[\left(\frac{\partial F}{\partial c_x} \right) \delta c \right] - \int \frac{d}{dx} \left(\frac{\partial F}{\partial c_x} \right) \delta c dx \right\} dy \tag{13}$$

$$= -\iint \left\{ \frac{d}{dx} \left(\frac{\partial F}{\partial c_x} \right) \delta c \right\} dxdy$$

$$\iint \left(\frac{\partial F}{\partial c_y} \right) \delta c_y dxdy = \int \left\{ \int \left(\frac{\partial F}{\partial c_y} \right) \delta c_y dy \right\} dx$$

$$= \int \left\{ \left[\left(\frac{\partial F}{\partial c_y} \right) \delta c \right] - \int \frac{d}{dy} \left(\frac{\partial F}{\partial c_y} \right) \delta c dy \right\} dx \tag{14}$$

$$= -\iint \left\{ \frac{d}{dy} \left(\frac{\partial F}{\partial c_y} \right) \delta c \right\} dxdy$$

したがって、

$$\delta J = \iint \left(\frac{\partial F}{\partial c} \right) \delta c dxdy + \iint \left(\frac{\partial F}{\partial c_x} \right) \delta c_x dxdy + \iint \left(\frac{\partial F}{\partial c_y} \right) \delta c_y dxdy$$

$$= \iint \left(\frac{\partial F}{\partial c} \right) \delta c dxdy - \iint \left\{ \frac{d}{dx} \left(\frac{\partial F}{\partial c_x} \right) \delta c \right\} dxdy - \iint \left\{ \frac{d}{dy} \left(\frac{\partial F}{\partial c_y} \right) \delta c \right\} dxdy \tag{15}$$

$$= \iint \left\{ \left(\frac{\partial F}{\partial c} \right) - \frac{d}{dx} \left(\frac{\partial F}{\partial c_x} \right) - \frac{d}{dy} \left(\frac{\partial F}{\partial c_y} \right) \right\} \delta c dxdy$$

これより、オイラー方程式は、

$$\left(\frac{\partial F}{\partial c} \right) - \frac{d}{dx} \left(\frac{\partial F}{\partial c_x} \right) - \frac{d}{dy} \left(\frac{\partial F}{\partial c_y} \right) = 0 \tag{16}$$

ここで、

F は c, x, y, c_x, c_y の関数であるので、x で微分する場合には合成関数として扱わなくてはならない。たとえば、

$$dF = \left(\frac{\partial F}{\partial x} \right)(dx) + \left(\frac{\partial F}{\partial y} \right)(dy) + \left(\frac{\partial F}{\partial c} \right)(dc) + \left(\frac{\partial F}{\partial c_x} \right)\{d(c_x)\} + \left(\frac{\partial F}{\partial c_y} \right)\{d(c_y)\}$$

$$\frac{dF}{dx} = \left(\frac{\partial F}{\partial x} \right) + \left(\frac{\partial F}{\partial c} \right)\left(\frac{dc}{dx} \right) + \left(\frac{\partial F}{\partial c_x} \right)\left(\frac{d^2 c}{dx^2} \right) + \left(\frac{\partial F}{\partial c_y} \right)\left(\frac{d^2 c}{dxdy} \right) \tag{17}$$

同様に

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{\partial F}{\partial c_x}\right) &= \left(\frac{\partial}{\partial x}\right)\left(\frac{\partial F}{\partial c_x}\right) + \left(\frac{\partial}{\partial c}\right)\left(\frac{\partial F}{\partial c_x}\right)\left(\frac{dc}{dx}\right) + \left(\frac{\partial}{\partial c_x}\right)\left(\frac{\partial F}{\partial c_x}\right)\left(\frac{d^2c}{dx^2}\right) + \left(\frac{\partial}{\partial c_y}\right)\left(\frac{\partial F}{\partial c_x}\right)\left(\frac{d^2c}{dxdy}\right) \\
&= \left(\frac{\partial^2 F}{\partial x \partial c_x}\right) + \left(\frac{\partial^2 F}{\partial c \partial c_x}\right)\left(\frac{dc}{dx}\right) + \left(\frac{\partial^2 F}{\partial c_x^2}\right)\left(\frac{d^2c}{dx^2}\right) + \left(\frac{\partial^2 F}{\partial c_y \partial c_x}\right)\left(\frac{d^2c}{dxdy}\right) \\
\frac{d}{dy}\left(\frac{\partial F}{\partial c_y}\right) &= \left(\frac{\partial}{\partial y}\right)\left(\frac{\partial F}{\partial c_y}\right) + \left(\frac{\partial}{\partial c}\right)\left(\frac{\partial F}{\partial c_y}\right)\left(\frac{dc}{dx}\right) + \left(\frac{\partial}{\partial c_x}\right)\left(\frac{\partial F}{\partial c_y}\right)\left(\frac{d^2c}{dxdy}\right) + \left(\frac{\partial}{\partial c_y}\right)\left(\frac{\partial F}{\partial c_y}\right)\left(\frac{d^2c}{dy^2}\right) \\
&= \left(\frac{\partial^2 F}{\partial y \partial c_y}\right) + \left(\frac{\partial^2 F}{\partial c \partial c_y}\right)\left(\frac{dc}{dy}\right) + \left(\frac{\partial^2 F}{\partial c_x \partial c_y}\right)\left(\frac{d^2c}{dxdy}\right) + \left(\frac{\partial^2 F}{\partial c_y^2}\right)\left(\frac{d^2c}{dy^2}\right)
\end{aligned}
\tag{18}
$$

である。したがって、最終的にオイラー方程式は次式にて与えられる。

$$
\begin{aligned}
&\left(\frac{\partial F}{\partial c}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial c_x}\right) - \frac{d}{dy}\left(\frac{\partial F}{\partial c_y}\right) \\
&= \left(\frac{\partial F}{\partial c}\right) - \left\{\left(\frac{\partial^2 F}{\partial x \partial c_x}\right) + \left(\frac{\partial^2 F}{\partial c \partial c_x}\right)\left(\frac{dc}{dx}\right) + \left(\frac{\partial^2 F}{\partial c_x^2}\right)\left(\frac{d^2c}{dx^2}\right) + \left(\frac{\partial^2 F}{\partial c_y \partial c_x}\right)\left(\frac{d^2c}{dxdy}\right)\right\} \\
&\quad - \left\{\left(\frac{\partial^2 F}{\partial y \partial c_y}\right) + \left(\frac{\partial^2 F}{\partial c \partial c_y}\right)\left(\frac{dc}{dy}\right) + \left(\frac{\partial^2 F}{\partial c_y^2}\right)\left(\frac{d^2c}{dy^2}\right) + \left(\frac{\partial^2 F}{\partial c_x \partial c_y}\right)\left(\frac{d^2c}{dxdy}\right)\right\} = 0
\end{aligned}
\tag{19}
$$